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The long-distance propagation of shallow water waves over an ocean of random depth

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An investigation is made of the scattering effect of a random ocean bottom of constant average depth upon the propagation of shallow water waves. Of particular concern is the case of long-distance propagation, in which the conventional perturbation schemes fail to apply. The approximation scheme employed is basically one of selective summation of the type used in other areas of physics such as the theory of many-body interactions. Results are obtained for the average wave and the two-point correlation function of the wave field for the case when the ocean statistics are homogeneous and isotropic. The application of the results to the case of a tsunami is discussed.

1. Introduction

An interesting and successful application of hydrodynamics is that to the propagation, under gravity, of disturbances on the free surface of a liquid. In particular, the linear 'shallow-water' theory predicts that long waves over an ocean of uniform depth travel at constant velocity without change in shape. In this paper we investigate the propagation of long waves, over large distances, in an ocean whose depth varies in a random or stochastic fashion about a constant mean value. The wave field is, therefore, governed by a linear partial differential equation with stochastic coefficients. It is well known that this type of equation, although linear in the field quantities, is nonlinear with respect to the stochastic variables. Thus an attempt to obtain a particular moment of the wave field results in the appearance of higher order moments, and an infinite hierarchy of simultaneous equations must be solved in order to obtain any single moment.

The problem of wave propagation in random media has received considerable attention in recent years (see, for example, Frisch 1968) and nearly all the approaches have used a perturbation analysis of one form or another. The main idea is to linearize the problem by expanding the wave field in terms of a small parameter characterizing the strength of the random inhomogenieties. Various methods of effecting this perturbation have been proposed.

For the case of long-distance propagation the conventional perturbation schemes fail to apply. Hence, although Keller (1958) has offered a mathematical explanation of the use of geometrical optics for a shallow-water variable-depth analysis, this approach is not applicable over long distances since it fails to

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account properly for the effects of diffraction. The Born approximation, on the other hand, is valid only for finite scattering regions. In addition it is a single scattering approximation. Although Kay & Silverman (1958) have suggested that the randomness of the medium reduces the importance of the multiple scattering terms, the cumulative effect of these terms over large distances cannot be neglected. Katz (1962) has applied both geometrical optics and the Born approximation to the scattering of long waves over a random ocean. His results, therefore, fail to hold for large distances. Kajiura (1961) has applied the Rytov method, or method of smooth perturbations, to the two-dimensional propagation of long waves over a random bottom. This technique, however, also fails at large distances as was demonstrated by Brown (1966).

The underlying difficulty with the conventional perturbation schemes is that they are of finite order. That is, the approximation in each case involves only a finite number of the terms in the original perturbation expansion. For homogeneous random media these terms are secular, i.e. proportional to the propagation distance, and consequently for large distances are no longer small, regardless of the strength of the inhomogeneities. However, in recent years considerable progress has been made in overcoming this difficulty; formal perturbation schemes that selectively sum infinite subsets of terms in the original expansion have been developed. In this paper a diagrammatic version of this type of technique is employed.

2. Governing equations

We consider the motion of an incompressible inviscid fluid of constant density over an ocean of variable depth $h(\mathbf{r})$, where $\mathbf{r} = (x, y)$. It is well known (see Stoker 1957) that the propagation of small amplitude long waves in such an ocean is governed by the equation

$$\partial^2\eta/\partial t^2 = g\nabla . (h\nabla\eta) + q, \tag{2.1}$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$, $\eta(\mathbf{r}, t)$ is the elevation at time t of the free surface above the horizontal plane $z = 0$, g is the acceleration of gravity, and

$$q(\mathbf{r}, t) = \frac{\partial}{\partial t}\int_{-h}^{\eta} Q(\mathbf{r}, z, t)\, dz,$$

with $Q(\mathbf{r}, z, t)$ a given volume source term. We shall assume that $h(\mathbf{r})$ is a statistically homogeneous random function and that the fluctuations of $h(\mathbf{r})$ about its mean value are small. Thus we let $\langle\,\rangle$ denote ensemble averages, write $h(\mathbf{r}) = \langle h\rangle + h'(\mathbf{r})$ and assume that $\epsilon \equiv \langle h'^2\rangle^{\frac{1}{2}}/\langle h\rangle \ll 1$. For this case, Katz (1962) has shown that in order for (2.1) to describe the surface elevation adequately the following conditions must be met: $\langle h\rangle/\lambda \ll 1$, where λ is a characteristic wavelength; $\eta_{\mathrm{rms}}/\langle h\rangle \ll \langle h'^2\rangle^{\frac{1}{2}}/\langle h\rangle$, where $\eta_{\mathrm{rms}} = \langle\eta^2\rangle^{\frac{1}{2}}$ is the root-mean-square surface elevation; $l \geqslant \lambda$, where l is the correlation length of the random function $h'(\mathbf{r})$. The first condition is the usual shallow-water assumption; the second ensures that it is the randomness of the bottom and not the neglected nonlinear terms in (2.1) which is the important perturbing disturbance to the wave field and the third ensures that the basic wave motion is similar to that over a flat bottom.

We shall consider the steady-state time-harmonic problem associated with (2.1) in which

$$Q(\mathbf{r},z,t) = Q_0(\mathbf{r})\,\delta(z-z_0)\exp(-i\omega t) \quad (-h(\mathbf{r}) < z_0 < \eta(\mathbf{r},t))$$

and $Q_0(\mathbf{r})$ vanishes outside some neighbourhood of $\mathbf{r} = 0$. If we let

$$\eta(\mathbf{r},t) = \phi(\mathbf{r})\exp(-i\omega t)$$

we find that ϕ satisfies

$$\nabla^2\phi + k_0^2\phi = -\epsilon\nabla.(\mu\nabla\phi)+f, \tag{2.2}$$

where $k_0^2 = \omega^2/g\langle h\rangle$, $\mu(\mathbf{r}) = h'(\mathbf{r})/[\langle h'^2(\mathbf{r})\rangle]^{\frac{1}{2}}$ and $f(\mathbf{r}) = -i(g\langle h\rangle)^{-1}\omega Q_0(\mathbf{r})$. It is perhaps worth noting that the physical process of wave generation is not important for our purposes since we shall examine the waves far from the source. Thus no particular physical significance is attached to Q. By assuming that $\phi(\mathbf{r})$ satisfies the radiation condition

$$\lim_{r\to\infty}\{|\mathbf{r}|^{\frac{1}{2}}(\partial\phi/\partial r - ik_0\phi)\} = 0 \tag{2.3}$$

and defining $G_0(\mathbf{r},\boldsymbol{\rho}) = \frac{1}{4}iH_0^{(1)}(k_0|\mathbf{r}-\boldsymbol{\rho}|)$, where $H_0^{(1)}$ is the zero-order Hankel function of the first kind, it can be shown that (2.2) is equivalent to the following integral equation for $\phi_{,i} = \partial\phi/\partial r_i\,(i = 1,2)$:

$$\phi_{,i}(\mathbf{r}) = \phi_{,i}^0(\mathbf{r}) + \epsilon[G_{0,ij}(\mathbf{r},\boldsymbol{\rho}),\mu(\boldsymbol{\rho})\,\phi_{,j}(\boldsymbol{\rho})]. \tag{2.4}$$

In the above equation (with summation over repeated indices implied) we have defined

$$\phi_{,i}^0(r) = \int\frac{\partial G_0(\mathbf{r},\boldsymbol{\rho})}{\partial r_i}f(\boldsymbol{\rho})\,d^2\boldsymbol{\rho} \tag{2.5}$$

and

$$[G_{0,ij}(\mathbf{r},\boldsymbol{\rho}),\psi_j] = \lim_{\delta\to 0}\int_{|\mathbf{r}-\boldsymbol{\rho}|\geqslant\delta}\frac{\partial^2G_0(\mathbf{r},\boldsymbol{\rho})}{\partial r_i\,\partial\rho_j}\,\psi_j(\boldsymbol{\rho})\,d^2\boldsymbol{\rho}, \tag{2.6}$$

where, in fact, it can be shown that

$$[G_{0,ij}(\mathbf{r},\boldsymbol{\rho}),\psi_j(\boldsymbol{\rho})] = -[G_{0,i}(\mathbf{r},\boldsymbol{\rho}),\psi_{j,j}(\boldsymbol{\rho})]. \tag{2.7}$$

We assume that

$$\phi_{,i} = \sum_{n=0}^{\infty}\epsilon^n\phi_{,i}^{(n)},$$

substitute this into (2.4), take averages, and find that

$$\langle\phi_{,i}^{(n)}(\mathbf{r})\rangle = [G_{0,ij_1}(\mathbf{r},\boldsymbol{\rho}_1),\ldots,[G_{0,j_{n-1}j_n}(\boldsymbol{\rho}_{n-1},\boldsymbol{\rho}_n),\langle\mu(\boldsymbol{\rho}_1)\ldots\mu(\boldsymbol{\rho}_n)\rangle\,\phi_{,j_n}^{(0)}(\boldsymbol{\rho}_n)]\ldots]. \tag{2.8}$$

In what follows, we shall assume that μ is a Gaussian random process. We shall show later that this is not a restrictive assumption for $\epsilon \ll 1$. In this case all odd moments of μ vanish and all moments containing $2n$ μ-factors yield $(2n)!/2^n n!$ terms, constructed from all possible permutations taken two at a time. Thus

$$\langle\phi_{,i}(\mathbf{r})\rangle = \sum_{\nu=0}^{\infty}\epsilon^{2\nu}\langle\phi_{,i}^{(2\nu)}(\mathbf{r})\rangle = \sum_{\nu=0}^{\infty}\epsilon^{\nu}\langle\phi_{,i}^{(\nu)}(\mathbf{r})\rangle,$$

and it is convenient to represent this series diagrammatically by using the symbols defined in figure 1. This representation is shown in figure 2. Of course, detailed rules may be given for construction of the diagram in figure 2, but these

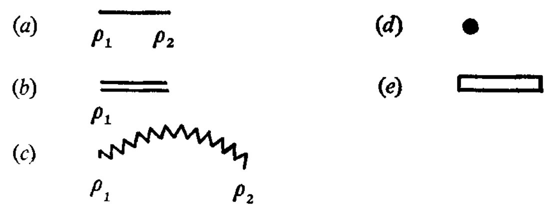

FIGURE 1. Diagrammatic symbols for (*a*) $G_{0,j_1j_2}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$, (*b*) $\phi^0_{,j_1}(\boldsymbol{\rho}_1)$, (*c*) $\langle\mu(\boldsymbol{\rho}_1)\,\mu(\boldsymbol{\rho}_2)\rangle$, (*d*) δ_{jk}, (*e*) $\langle\phi_{,i}(\mathbf{r})\rangle$.

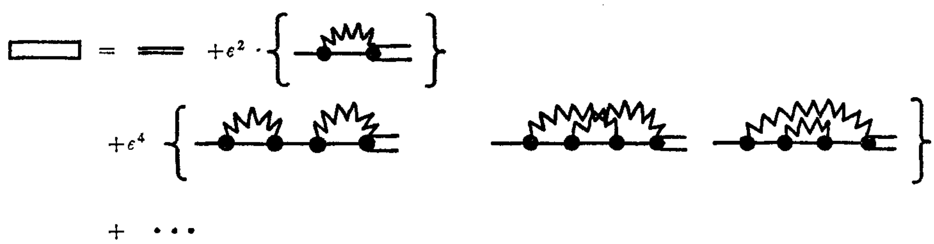

FIGURE 2. Diagrammatic representation of $\langle\phi_{,i}(\mathbf{r})\rangle$ showing terms to fourth order in ϵ.

are fairly obvious if we note, for example, that the fourth term on the right-hand side of the equality sign in figure 2 represents the term

$$[G_{0,ij_1}(\mathbf{r}, \boldsymbol{\rho}_1), [G_{0,j_1j_2}, [G_{0,j_2j_3}, [G_{0,j_3j_4}(\boldsymbol{\rho}_3, \boldsymbol{\rho}_4), \langle\mu(\boldsymbol{\rho}_1)\,\mu(\boldsymbol{\rho}_3)\rangle\,\langle\mu(\boldsymbol{\rho}_2)\,\mu(\boldsymbol{\rho}_4)\rangle\,\phi^0_{,j_4}(\boldsymbol{\rho}_4)]]]]. \quad (2.9)$$

An integral equation for $\langle\phi_{,i}(\mathbf{r})\rangle$ may be obtained from figure 2 by using the following summation procedure. First, all double lines ($\phi^0_{,j_n}$'s) appearing in figure 2 are replaced by single lines ($G_{0,j_{n-1}j_n}$'s). Second, the diagrams which are modified by the first step are used to define the effective Green's function $G_{e,ij}(\mathbf{r}, \boldsymbol{\rho})$, as shown in figures 3(*a*), (*b*) and (*c*). Third, the effective Green's function is used to reduce figure 2 to the integral equation shown in figure 3(*d*). Analytically, we have

$$\langle\phi_{,i}(\mathbf{r})\rangle = \phi^0_{,i}(\mathbf{r}) + \epsilon^2[G_{0,ij_1}(\mathbf{r}, \boldsymbol{\rho}_1), [K^{(1)}_{j_1j_2}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2), \langle\phi_{,j_2}(\boldsymbol{\rho}_2)\rangle]], \quad (2.10)$$

where $K^{(1)}_{j_1j_2}$ is defined as in figure 3(*c*).

Using the diagrammatic techniques we have outlined, it is not hard to find an integral equation for the second moment of the wave field. From our assumption as to the Gaussian nature of the process $\mu(\mathbf{r})$, it is clear that this moment will be given by the infinite series

$$\langle\phi_{,i}(\mathbf{r})\,\phi^*_{,j}(\mathbf{r}')\rangle = \sum_{n=0}^{\infty} \epsilon^{2n} \sum_{k+l=n} \langle\phi^{(k)}_{,i}(\mathbf{r})\,\phi^{(l)*}_{,j}(\mathbf{r}')\rangle,$$

where * denotes complex conjugate. We introduce the symbol for $\langle\phi_{,i}(\mathbf{r})\phi^*_{,j}(\mathbf{r}')\rangle$ shown in figure 4(*a*), use (2.4) to calculate each term in the series, and construct diagrammatic representations of the terms using the same rules as for the first moment, except that products are denoted by placing one diagram above the other. The result is shown in figure 4(*b*), where, for example, the second diagram

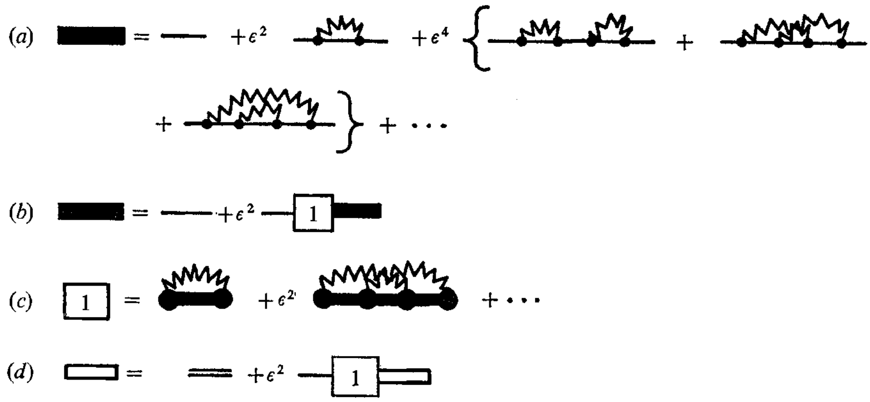

FIGURE 3. Diagrams showing (*a*) the definition of $G_{e,ij}(\mathbf{r},\boldsymbol{\rho})$, (*b*) the integral equation for $G_{e,ij}(\mathbf{r},\boldsymbol{\rho})$, (*c*) the definition of $K^{(1)}_{j_1j_2}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$, (*d*) the integral equation for $\langle\phi_{,i}(\mathbf{r})\rangle$.

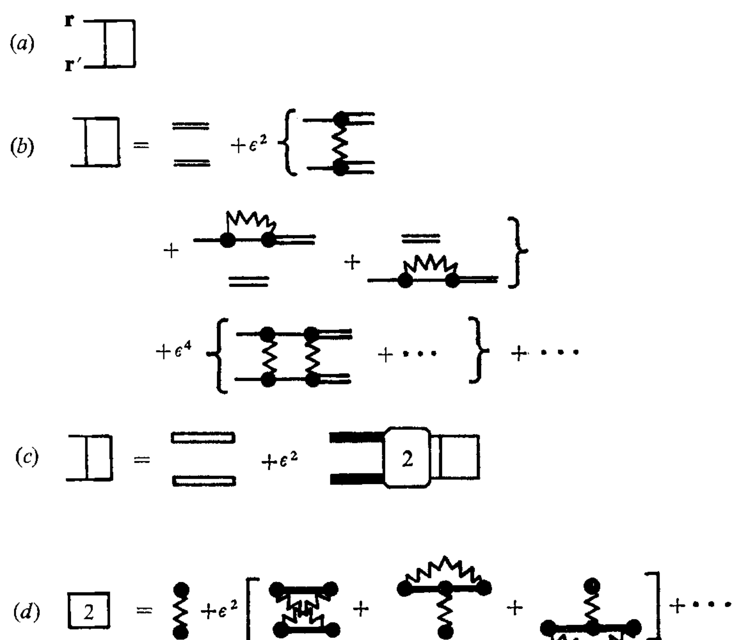

FIGURE 4. Diagrammatic representation for (*a*) $\langle\phi_{,i}(\mathbf{r})\,\phi^*_{,j}(\mathbf{r})'\rangle$, (*b*) the series

$$\sum_{n=0}^{\infty}\epsilon^{2n}\sum_{k+l=2n}\langle\phi^{(k)}_{,i}(\mathbf{r})\,\phi^{(l)}_{,j}(\mathbf{r}')\rangle,$$

(*c*) equation (2.12), (*d*) $K^{(2)}_{j_1j_2j_3j_4}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\boldsymbol{\rho}_3,\boldsymbol{\rho}_4)$.

on the right-hand side of the equality sign in figure 4(*b*) represents the term

$$[G_{0,ij_1}(\mathbf{r},\boldsymbol{\rho}_1),[G^*_{0,jj_2}(\mathbf{r}',\boldsymbol{\rho}_2),\langle\mu(\boldsymbol{\rho}_1)\,\mu(\boldsymbol{\rho}_2)\rangle\,\phi^0_{,j_1}(\boldsymbol{\rho}_1)\,\phi^{0*}_{,j_2}(\boldsymbol{\rho}_2)]]. \tag{2.11}$$

The sum of all diagrams whose top and bottom lines are not connected by wavy lines is $\langle\phi_{,i}(\mathbf{r})\rangle\langle\phi^*_{,j}(\mathbf{r}')\rangle$, as may be seen by comparison with figure 2. The remaining diagrams can be summed by introducing the effective Green's function.

After some manipulation, we find that figure 4(*b*) is equivalent to the diagrammatic equation in figure 4(*c*). Thus, the second moment satisfies the integral equation

$$\langle \phi_{,i}(\mathbf{r})\, \phi^*_{,j}(\mathbf{r}')\rangle = \langle \phi_{,i}(\mathbf{r})\rangle \langle \phi^*_{,j}(\mathbf{r}')\rangle + \epsilon^2[G_{e,ij_1}(\mathbf{r}, \boldsymbol{\rho}_1), [G^*_{e,jj_2}(\mathbf{r}', \boldsymbol{\rho}_2), K^{(2)}_{j_1j_2j_3j_4}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_3, \boldsymbol{\rho}_4)\langle \phi_{,j_3}(\boldsymbol{\rho}_3)\, \phi^*_{,j_4}(\boldsymbol{\rho}_4)\rangle]], \quad (2.12)$$

where $K^{(2)}_{j_1j_2j_3j_4}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_3, \boldsymbol{\rho}_4)$ is defined in figure 4(*d*).

3. Perturbation solutions for the first and second moments

As they stand, (2.10) and (2.12) are non-tractable since their kernels are themselves infinite series. These equations are important, however, because they clearly indicate the nature of any approximations one would make in order to render the equations solvable.

An examination of the behaviour of the terms in (2.10) reveals that for distances of propagation, x, large compared with the correlation length l of the bottom fluctuations, diagrams of order ϵ^{2n} are secular or increase with propagation distance as $(x/l)^{2n-m}$, where m is the number of times that severing each interior straight line in a diagram results in the severing of one or more wavy lines. Therefore $n \leqslant m \leqslant 2n-1$, and it follows that the diagram for which $m = n$ is larger than the other diagrams of order ϵ^{2n} by at least x/l, which is very much greater than one. It would be tempting at this point to disregard diagrams of order ϵ^{2n} for which $m \neq n$. However, since diagrams of lower order in ϵ are secular to the same degree as diagrams of higher order, it becomes necessary to compare terms with the same degree of secularity. When lengths are made dimensionless by using the correlation length l, it is found that terms for which $m \neq n$ may be neglected for large propagation distances provided that the condition $\epsilon^2k^2l^2 \ll 1$ is met. Inspection of the resulting equation indicates that omitting terms for which $m \neq n$ is equivalent to approximating K_1 by the first term on the right-hand side of the equality sign in figure 3(*c*) with $G_e = G_0$. This type of approximation has been called the smoothing method and its validity has been the subject of much investigation (see Frisch (1968), and the references cited therein, and Molyneux (1971) for a discussion of the effects of keeping higher order terms in figure 3(*c*)). The complexity of the equations precludes its rigorous justification, but in certain cases it is possible to compare the smoothing method with the exact solution for solvable models (see McKenna & Morrison 1970). It is found that the approximate and exact solutions match closely in these cases. Therefore, let us write $\Phi_{1,i}(\mathbf{r}) = \langle \phi_{,i}(\mathbf{r})\rangle$ and assume that

$$\Phi_{1,i}(\mathbf{r}) = \phi^0_{,i}(\mathbf{r}) + \epsilon^2[G_{0,ij_1}(\mathbf{r}, \boldsymbol{\rho}_1), [G_{0,j_1j_2}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2), \langle \mu(\boldsymbol{\rho}_1)\, \mu(\boldsymbol{\rho}_2)\rangle\, \Phi_{1,j_2}(\boldsymbol{\rho}_2)]]. \quad (3.1)$$

In addition, we shall assume that μ is statistically isotropic as well as homogeneous, so that

$$\langle \mu(\mathbf{r})\, \mu(\mathbf{r}')\rangle = R(|\mathbf{r}-\mathbf{r}'|).$$

By employing property (2.7) it can be shown that (3.1) is equivalent to

$$\Phi_1(\mathbf{r}) = \phi^0(\mathbf{r}) + \epsilon^2 \int G_0(\mathbf{r}, \boldsymbol{\rho}_1) \frac{\partial}{\partial \boldsymbol{\rho}_{1j_1}} \int \frac{\partial G_0}{\partial \boldsymbol{\rho}_{1j_1}}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \frac{\partial}{\partial \boldsymbol{\rho}_{2j_2}} \left(R(|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|) \frac{\partial \Phi_1}{\partial \boldsymbol{\rho}_{2j_2}}\right) d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2. \quad (3.2)$$

If we define the convolution of the vectors $\mathbf{u}(\mathbf{r})$, $\mathbf{v}(\mathbf{r})$ by

$$\mathbf{u}*\mathbf{v} = \frac{1}{2\pi}\int \mathbf{v}(\boldsymbol{\rho})\,\mathbf{u}(\mathbf{r}-\boldsymbol{\rho})\,d^2\boldsymbol{\rho}, \tag{3.3}$$

then Φ_1 satisfies an integral equation of the convolution type and can be determined by Fourier transformation. Defining

$$\mathscr{F}(\psi) = \frac{1}{2\pi}\int e^{i\mathbf{k}.\mathbf{r}}\,\psi(\mathbf{r})\,d^2\mathbf{r} = \bar{\psi}(\boldsymbol{\kappa}) \tag{3.4}$$

and applying the operator $\mathscr{F}$ to both sides of (3.2) we find that

$$\bar{\Phi}_1(\boldsymbol{\kappa}) = \bar{f}(\boldsymbol{\kappa})\,D^{-1}(\kappa), \tag{3.5}$$

where $\kappa = |\boldsymbol{\kappa}|$,
$$D(\boldsymbol{\kappa}) = k_0^2 - \kappa^2 - \epsilon^2\kappa^2\bar{Q}(\kappa) \tag{3.6}$$

and
$$\bar{Q}(\kappa) = i\kappa_{j_1}\overline{G_{0,j_1}R}(\boldsymbol{\kappa}) - \kappa_{j_1}\kappa_{j_2}\kappa^{-2}\,\overline{G_{0,j_1}R_{,j_2}}(\boldsymbol{\kappa}). \tag{3.7}$$

Defining an effective wavenumber k_e as that value of κ satisfying $D(\kappa) = 0$, it then follows that

$$\langle\phi(\mathbf{r})\rangle = \Phi_1(\mathbf{r}) = F(k_e)\int G_e(\mathbf{r},\boldsymbol{\rho})f(\boldsymbol{\rho})\,d^2\boldsymbol{\rho}, \tag{3.8}$$

where $G_e(\mathbf{r},\boldsymbol{\rho})$, the effective Green's function, is given by the free-space Green's function with k_0 replaced by k_e, and where

$$F(k_e) = [1+\epsilon^2\bar{Q}(k_e) + \tfrac{1}{2}\epsilon^2k_0^2\bar{Q}'(k_e)]^{-1}. \tag{3.9}$$

The solution for the average coherent wave thus depends upon the properties of k_e, which in turn depend upon the particular form for $R(\mathbf{r})$. It should be noted that since derivatives of μ appear in the expression for k_e, the form

$$R(r) = \exp(-r/l)$$

is not permissible, since for this choice $\mu(\mathbf{r})$ is not even mean-square differentiable. Approximate values of k_e are calculated in the appendix for the case when

$$R(r) = \exp(-r^2/l^2). \tag{3.10}$$

When $\epsilon^2k_0^2l^2 \ll 1$, so that the smoothing approximation is justified, the result is that

$$k_e = k_0 + \tfrac{1}{2}\epsilon^2k_0(1-1/2\gamma) + \tfrac{1}{4}\epsilon^2\gamma k_0.\exp(-\gamma)\,(K_0(\gamma)+K_1(\gamma)/\gamma) + \tfrac{1}{4}i\pi k_0\epsilon^2\gamma\exp(-\gamma)\,(I_0(\gamma)-I_1(\gamma)/\gamma), \tag{3.11}$$

where $\gamma = \frac{1}{2}k_0^2l^2$ and I_n and K_n are the modified Bessel functions (of order $n = 0, 1$) of the first and second kinds respectively. In view of the fact that k_e has a positive imaginary part, the amplitude of the coherent wave decreases with increasing propagation distance. Physically this is to be expected since the randomness of the ocean bottom has the effect of continually scattering energy from the coherent wave to the fluctuating, or incoherent portion of the wave field.

It is of interest to consider what effect the $\nabla\mu.\nabla\phi$ term in (2.2) has upon the effective wavenumber. When considering the case when k_0l is large it is customary to neglect this term and to consider the equation

$$(1+\epsilon\mu)\,\nabla^2\phi + k_0^2\phi = f(\mathbf{r}). \tag{3.12}$$

By proceeding in a fashion similar to that above it can be shown that when the correlation function is again given by (3.10) the expression for the effective wavenumber (designated in this case by $\tilde{k}_e$) becomes

$$\tilde{k}_e = k_0 + \tfrac{1}{2}\epsilon^2 k_0 + \tfrac{1}{4}\epsilon^2 k_0 \gamma \exp(-\gamma)\,(K_0(\gamma) + i\pi I_0(\gamma)), \tag{3.13}$$

and by comparing (3.11) and (3.13) it is seen that for $k_0 l \gg 1$ the expressions for k_e and $\tilde{k}_e$ become equivalent, as expected. It is interesting to note, however, that omitting the $\nabla\mu.\nabla\phi$ term results in an effective wavenumber with a larger imaginary part than that which would have been obtained if this term had been retained. On the other hand, it can be shown that omitting the $\nabla\mu.\nabla\phi$ term does not alter the effective wavenumber for a one-dimensional ocean, where the scattering is restricted to the forward and backward directions. Consequently, it can be inferred that the $\nabla\mu.\nabla\phi$ term in the two-dimensional case tends to reduce scattering outside the direction of the coherent wave. It follows that, for a fixed propagation distance, increasing the correlation length of an isotropic bottom decreases the off-axis scattering, which results in an increase in the amplitude of the coherent wave.

Let us now turn to the calculation of the second moment

$$\Phi_2(\mathbf{r},\mathbf{r}') = \langle \phi(\mathbf{r})\,\phi^*(\mathbf{r}')\rangle.$$

In the smoothing approximation we again replace $K^{(2)}_{j_1 j_2 j_3 j_4}$ in (2.12) by the first term on the right-hand side of the equals sign in figure 4(*d*). We then have

$$\Phi_{2,ij}(\mathbf{r},\mathbf{r}') = \Phi_{1,i}(\mathbf{r})\,\Phi^*_{1,j}(\mathbf{r}') + \epsilon^2[G_{e,ij_1}(\mathbf{r},\boldsymbol{\rho}_1),[G^*_{e,jj_2}(\mathbf{r}',\boldsymbol{\rho}_2), R(|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|)\,\Phi_{2,j_1j_2}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)]]. \tag{3.14}$$

By again employing property (2.7) it can be shown that (3.14) is equivalent to

$$\Phi_2(\mathbf{r},\mathbf{r}') = \Phi_1(\mathbf{r})\,\Phi_1^*(\mathbf{r}') + \epsilon^2 \iint G_e(\mathbf{r},\boldsymbol{\rho}_1)\,G_e^*(\mathbf{r}',\boldsymbol{\rho}_2)\,\{R(|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|)\,\Phi_{2,j_1j_2}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)\}_{,j_1j_2}\,d^2\boldsymbol{\rho}_1 d^2\boldsymbol{\rho}_2. \tag{3.15}$$

We shall obtain an approximate solution of (3.15) for the case in which the source produces an average wave which depends only on x, that is, an attenuated plane wave. Then the second moment may be assumed to be of the form

$$\Phi_2(\mathbf{r},\mathbf{r}') = \Phi_2(x, x'-x, |y'-y|), \tag{3.16}$$

which is consistent with the assumption that μ is statistically homogeneous and isotropic. In our calculation of Φ_2 we shall assume that back-scattering is negligible and that $kl \gg 1$. In this case only small-angle scattering is important and we may neglect the $\nabla\mu.\nabla$ terms in (3.15) on the basis of our arguments above. We have, for $x = x' > 0$,

$$\Phi_2(x,0,|y'-y|) = \Phi_1(x)\,\Phi_1^*(x) + \epsilon^2 k_0^4 \iint G_e(\mathbf{r},\boldsymbol{\rho}_1)\,G_e^*(\mathbf{r}',\boldsymbol{\rho}_2)\,R(|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|) \times \Phi_2(x_1, x_2-x_1, |y_2-y_1|)\,d^2\boldsymbol{\rho}_1 d^2\boldsymbol{\rho}_2, \tag{3.17}$$

where $\boldsymbol{\rho}_j = (x_j, y_j)$. The integral in (3.17) may be simplified considerably. First, since we are considering long distances of propagation, we may replace the Green functions by their asymptotic form

$$G_e(\mathbf{r},\boldsymbol{\rho}) \sim 4^{-1}(2\pi)^{-\frac{1}{2}}|\mathbf{r}-\boldsymbol{\rho}|^{-\frac{1}{2}}\exp[ik_e|\mathbf{r}-\boldsymbol{\rho}| + \tfrac{1}{4}i\pi].$$

Second, since only small-angle scattering is important, the major contribution to the integral comes from those regions included in a triangle with vertex at the observation point, with axis parallel to the x direction and with angular aperture $\theta = (k_0 l)^{-1} \ll 1$. In these regions, $k_e|\mathbf{r}-\boldsymbol{\rho}_1|$ in the exponent may be approximated by $k_e(x-x_1)+k_e(y-y_1)^2 \times \frac{1}{2}(x-x_1)^{-1}$, with an error of order $k_0 x(kl)^{-4}$, and we shall assume in what follows that $x < (k_0 l)^3 l$. Finally, keeping only the first term in the expansion of the denominator and changing variables from x_i and y_i to $x_i = s_i$ and $y_i - x_i = p_i$, the integral (I say) in (3.17) becomes

$$I = \frac{\epsilon^2 k_0^4}{16}\left(\frac{2}{\pi}\right)\int_{s_1}\int_{\mathbf{p}} \frac{\exp[ik_e(x-s_1)-ik_e^*(x-s_1-p_1)]}{[k_e(x-s_1)\,k_e^*(x-s_1-p_1)]^{\frac{1}{2}}} R(|\mathbf{p}|)\,\Phi_2(s_1,p_1,|p_2|)\,ds_1 d\mathbf{p}$$
$$\times \int_{s_2} \exp\left(\frac{ik_e(y-s_2)^2}{2(x-s_1)} - \frac{ik_e^*(y'-s_2-p_2)^2}{2(x-s_1-p_1)}\right) ds_2. \quad (3.18)$$

In the small-angle scattering approximation the term

$$-ik_e^*(y'-s_2-p_2)^2/2(x-s_1-p_1)$$

may be replaced by $-ik_e^*(y'-s_2-p_2)^2/2(x-s_1)$ since the neglected term is of the order $(k_0 l)^{-1}(p_1 l^{-1})$, which is very much less than one. In addition, in the denominator $x-s_1-p_1$ may be approximated by $x-s_1$. Integrating the resulting equation over s_2 yields

$$I \simeq \frac{\epsilon^2 k_0^4}{16}\left(\frac{2}{\pi}\right)\int_{s_1}\int_{\mathbf{p}} \exp[ik_e(x-s_1)]\exp[-ik_e^*(x-s_1-p_1)]\,R(|\mathbf{p}|)\,\Phi_2(s_1,p_1,|p_2|)$$
$$\times \left(\frac{2\pi}{A_1+A_2}\right)^{\frac{1}{2}} \exp\left[-\frac{(y'-y-p_2)^2}{2(A_1+A_2)}\right] ds_1\,dp_1\,dp_2, \quad (3.19)$$

where $A_1 = i(x-s_1)/k_e = A_2^*$. The integral over p_2 is of the form

$$\int_{-\infty}^{\infty} f(p_2)\exp[-(y'-y-p_2)^2\nu]\,dp_2,$$

where $\nu = (2(A_1+A_2))^{-1}$. Since ν is large and positive when $xl^{-1}\epsilon^2 \ll 1$, this integral may be approximated by using the Laplace method (e.g. Copson 1965), with the result that

$$\Phi_2(x,0,|y-y'|) = \Phi_1(x)\,\Phi_1^*(x) + \frac{\epsilon^2 k_0^2}{4}\int_{s_1=0}^{x}\int_{p_1} \exp[ik_e(x-s_1)$$
$$-ik_e^*(x-s_1-p_1)]\,\Phi_2(s_1,p_1,|y-y'|)\,R[(p_1^2+(y-y')^2)^{\frac{1}{2}}]\,ds_1\,dp_1. \quad (3.20)$$

Finally, it is assumed that

$$\Phi_2(s_1,p_1,|y-y'|) = \exp(-ik_0 p_1)\,\Phi_2(s,0,|y-y'|). \quad (3.21)$$

This form for the p_1 dependence is permissible since only values of $p_1 \leqslant l$ contribute significantly to the integral and at this distance significant diffraction does not occur. By the same reasoning, the term $(ik_e^* - ik_0)\,p_1$ is of order $\epsilon^2 k_0^2 l^2 (p_1 l^{-1})$, which is very much less than one, and may be neglected. Thus, (3.20), with $d = |y-y'|$, becomes

$$\Phi_2(x,0,d) = |\Phi_1(x)|^2 + \frac{\epsilon^2 k_0^2}{4}\int_0^x \exp[(ik_e - ik_e^*)(x-s_1)]\,\Phi_2(s,0,d)\,ds_1 F(d), \quad (3.22)$$

where

$$F(d) = \int_{-\infty}^{\infty} R[(p^2+d^2)^{\frac{1}{2}}]\,dp. \quad (3.23)$$

By substituting $\Phi_1(x) = \exp(ik_e x)$ and solving (3.22) for Φ_2, we find that

$$\Phi_2(x, 0, d) = |\Phi_1(x)|^2 \exp\{\tfrac{1}{4}\epsilon^2 k_0^2 F(d)\}. \tag{3.24}$$

This expression can be further simplified by noting from (A 4) (see appendix) that when $k_0 l \gg 1$

$$\operatorname{Im}(k_e) \simeq \frac{\epsilon^2 k_0^3 \pi}{4} \int_0^\infty J_0(k_0 r) J_0(k_0 r) R(r) r\, dr. \tag{3.25}$$

Since $R(r)r$ attains its maximum value for $r \simeq l$, the Bessel functions may be replaced by their asymptotic forms to obtain

$$\operatorname{Im}(k_e) \simeq \tfrac{1}{8}\epsilon^2 k_0^2 F(0) + O(\epsilon^2 k_0), \tag{3.26}$$

so that, restricting x such that $\epsilon^2 k_0 l(x/l) \ll 1$, it follows that

$$\Phi_2(x, 0, d) = \exp\{\tfrac{1}{4}\epsilon^2 k_0^2 (F(d) - F(0))\}. \tag{3.27}$$

Since $F(d) \leqslant F(0)$ the above expression indicates a loss of coherence as a result of the propagation over the random ocean. We note that (3.27) gives Φ_2 only for the case in which both observation points lie in the same plane ($x = x'$). We have not considered the case $x \neq x'$.

4. Discussion and application of results

It has been found that the long-distance propagation of shallow water waves is characterized by an effective wavenumber which governs the speed and attenuation of the coherent wave. Likewise the loss of coherence of the two-point correlation function was also found to be dependent upon the effective wavenumber. It is of interest to examine the conditions under which the above results apply to the propagation of tsunamis.

According to Cagle (1962), a typical tsunami is about 1 ft high, 100 miles long and travels at a speed of 250 miles per hour. The ocean is typically 12 000 ft deep, so that the shallow-water assumption $\lambda\langle h\rangle^{-1} \gg 1$ is clearly satisfied. Statistical information regarding the ocean floor is, on the other hand, rather limited. In the Pacific, where approximately four-fifths of all tsunamis occur, there exist large-scale features (such as the crest of the East Pacific Rise, large fracture zones, etc.) which violate our assumption that the depth is statistically homogeneous and isotropic. The analysis can therefore be applied only to those regions which exclude such topographic forms. Smith (1965) has performed a statistical analysis of the sea-floor relief in the north-east Pacific. The major conclusion to be drawn from this work is that, excluding the above mentioned anomalous regions, the depth distribution is nearly Gaussian for bottom features having wavelengths up to 16 miles, with the r.m.s. depth variation typically around 34 fathoms, so that $\epsilon \simeq 1{\cdot}7 \times 10^{-2}$. For these values the linear shallow-water equations hold, and the smoothing approximation is justified. Katz (1962), on the other hand, takes $\epsilon^2 \simeq 2 \times 10^{-2}$ and a correlation length of 500 miles. Although these values satisfy the linear theory they violate the smoothing requirement that $\epsilon^2 k^2 l^2 \ll 1$. Katz's values, however, were based on raw data taken over the Pacific for which the average depth was approximately 1400 fathoms. His value of ϵ

therefore corresponds to an r.m.s. depth variation of approximately 200 fathoms, a value representative of the large-scale features mentioned above.

Admittedly then, our theory fails to account properly for the larger scales over those localized portions of the ocean for which the smoothing requirement $\epsilon^2 k_0^2 l^2 \ll 1$ is violated. However, if we assume that their cumulative effect over long distances of propagation is small, we can determine those values of the r.m.s. depth variation and correlation length which are appropriate for the application of our theory. We shall try to infer these from an examination of the wave field itself. Tsunamis are known to travel great distances: for example, the Chilean tsunami of 23 May 1960 resulted in extensive damage to the Hawaiian Islands, some 6000 miles away. We shall neglect the effects of the earth's curvature and assume that scattering by irregularities in the ocean bottom attenuates the coherent wave by e^{-1} of its initial value after it has traversed 6000 miles. From (3.11) we see that $\mathrm{Im}\,(k_e/k_0) = (\epsilon k_0 l)^2 F(k_0 l)$, where

$$F(x) = \tfrac{1}{8}\pi \exp\left(-\tfrac{1}{2}x^2\right)\left[I_0(\tfrac{1}{2}x^2) - (2/x^2)\, I_1(\tfrac{1}{2}x^2)\right].$$

Since k_e is now known we can determine ϵ and l from (3.11) provided we assume that conditions ideal for the application of our theory prevail, i.e. provided we look for the minimum value of the perturbation parameter

$$(\epsilon k_0 l)^2 = \mathrm{Im}\,(k_e/k_0)/F(k_0 l)$$

in the region $k_0 l \gg 1$. We find that $(\epsilon k_0 l)^2$ is a minimum for $k_0 l \simeq 4{\cdot}2$, corresponding to a correlation length of 67 miles and a r.m.s. depth variation of 110 fathoms. These values lie between those of Smith and Katz, and in addition satisfy all of the above requirements.

Appendix

From (3.6) it follows that the effective wavenumber k_e satisfies the equation

$$k_e^2 = k_0^2 - \epsilon^2 k_e^2\, \bar{Q}(k_e), \tag{A 1}$$

where
$$\bar{Q}(k_e) = ik_{ej_1}\, \overline{G_{0,j_1} R}(k_e) - k_{ej_1} k_{ej_2} k_e^{-2}\, \overline{G_{0,j_1} R_{,j_2}}(k_e). \tag{A 2}$$

By introducing the relations

$$G_0(k_0 r) = -\tfrac{1}{4} i H_0^{(1)}(k_0 r), \quad G_{0,j_1} = \tfrac{1}{4} i k_0 H_1^{(1)}(k_0 r)\, r_{j_1} r^{-1}$$

and noting that

$$J_n(kr) = (2\pi i)^{-n} \int_0^{2\pi} \exp\left(ikr\cos\theta\right)\cos\left(n\theta\right) d\theta,$$

it is easy to show that

$$\begin{aligned} \bar{Q}(k_e) = &-\tfrac{1}{2} i\pi k_0^2 \int_0^\infty H_0^{(1)}(k_0 r)\, R(r)\, J_0(k_e r)\, r\, dr - 1 \\ &-\tfrac{1}{2} i\pi k_0^2 \int_0^\infty \{J_0(k_e r)\, H_1^{(1)}(k_0 r)\, k_0^{-1} + J_1(k_e r)\, H_0^{(1)}(k_0 r)\, k_e^{-1}\}\, R'(r)\, r\, dr \\ &-\tfrac{1}{2} i\pi k_0 k_e^{-1} \int_0^\infty H_1^{(1)}(k_0 r)\, J_1(k_e r)\, R''(r)\, r\, dr. \end{aligned} \tag{A 3}$$

The integrals above, where $R(r) = \exp(-r^2/l^2)$, are not expressible in closed form since the cylinder functions appear in combinations with those with different arguments and different orders. In order to obtain a closed-form expression for k_e the calculations will be limited to the case in which $\bar{Q}(k_e)$ may be approximated by $\bar{Q}(k_0)$. To within the same approximation k_e may be replaced by k_0. The second integral term in (A 3), I_2 say, may then be expressed in terms of cylinder functions of the same order by first noting that

$$I_2 = \tfrac{1}{2}i\pi \int_0^\infty R'(r)\{H_0^{(1)\prime}(k_0 r)J_0(k_0 r) + H_0^{(1)}(k_0 r)J_0'(k_0 r)\}k_0 r\,dr, \qquad (\text{A }4)$$

where

$$H_0^{(1)\prime}(z) = dH_0^{(1)}/dz = -H_1^{(1)}(z),$$

$$J_0'(z) = dJ_0/dz = -J_1(z)$$

and $z = k_0 r$. For $r > 0$, $J_0(z)$ and $H_0^{(1)}(z)$ satisfy the equations

$$z^2 H_0^{(1)\prime\prime}(z) + zH_0^{(1)\prime}(z) + z^2 H_0^{(1)}(z) = 0,$$

$$z^2 J_0''(z) + zJ_0'(z) + z^2 J_0(z) = 0.$$

Multiplying the first term by J_0/z, the second by $H_0^{(1)}/z$ and adding gives

$$d\{k_0 r(H_0^{(1)}J_0' + H_0^{(1)\prime}J_0)\}/dr = -2k_0^2 r(H_0^{(1)}J_0 - H_0^{(1)}J_1).$$

It follows that

$$\lim_{\delta\to 0}\int_\delta^r \frac{d}{d\xi}\{k_0\xi(H_0^{(1)}J_0' + H_0^{(1)\prime}J_0)\}\,d\xi = -2k_0^2 \lim_{\delta\to 0}\int_\delta^r \xi(H_0^{(1)}J_0 - H_1^{(1)}J_1)\,d\xi.$$

By introducing the asymptotic forms of $H_0^{(1)}(k_0\delta)$ and $J_0(k_0\delta)$, for $\delta \ll 1$ it can be shown that

$$k_0 r[H_0^{(1)}J_0' + H_0^{(1)\prime}J_0] = 2i\pi^{-1} - 2k_0^2\int_0^r \xi(H_0^{(1)}J_0 - H_1^{(1)}J_1)\,d\xi. \qquad (\text{A }5)$$

Substituting (A 5) into (A 4) and integrating the second term by parts yields

$$I_2 = i\pi k_0^2\int_0^\infty (H_0^{(1)}(k_0 r)J_0(k_0 r) - H_1^{(1)}(k_0 r)J_1(k_0 r))R(r)\,r\,dr + 1. \qquad (\text{A }6)$$

Each of the integrals in the resulting expression for $\bar{Q}(k_0)$ can then be written in the form

$$\int_0^\infty H_\nu^{(1)}(k_0 r)J_\nu(k_0 r)\exp(-r^2/l^2)\,r^{2\lambda+1}\,dr, \qquad (\text{A }7)$$

where λ is either 0 or 1. Consequently it suffices to consider the evaluation of integrals containing the products $J_\nu(k_0 r)J_\nu(k_0 r)$ and $J_\nu(k_0 r)J_{-\nu}(k_0 r)$. Now the product of two Bessel functions can be represented by the series (see, for example, Nielsen 1904, p. 25)

$$J_\nu(\alpha r)J_\rho(\beta r) = \alpha^\nu\beta^\rho\sum_{s=0}^\infty(-1)^s A_{\nu,\rho,s}(\alpha,\beta)\,(\tfrac{1}{2}r)^{\nu+\rho+2s}, \qquad (\text{A }8)$$

where

$$A_{\nu,\rho,s}(\alpha,\beta) = \frac{\alpha^{2s}}{s!\,\Gamma(\rho+1)\,\Gamma(\nu+s+1)}\,{}_2F_1(-\nu-s, -s; \rho+1; \beta^2/\alpha^2). \qquad (\text{A }9)$$

When $\alpha = \beta$ the hypergeometric function ${}_2F_1$ takes the simple form

$${}_2F_1(a,b;c;1) = \Gamma(c)\,\Gamma(c-a-b)/\Gamma(c-a)\,\Gamma(c-b), \qquad (\text{A }10)$$

which is exactly the simplification which was sought by assuming that $\bar{Q}(k_e) \simeq \bar{Q}(k_0)$. Taking $\alpha = \beta = k_0$, $\rho = \nu$ and integrating term-by-term yields

$$\int_0^\infty J_\nu(k_0 r) J_\nu(k_0 r) \exp(-r^2/l^2) r^{2\lambda+1} dr = \frac{l^{2(\lambda+1)}(2x)^\nu(-i)^\nu}{2\sqrt{\pi}} \sum_{s=0}^\infty \frac{(2xi)^s \Gamma(\nu+s+\frac{1}{2}) \Gamma(\nu+\lambda+s+1)}{s!\,\Gamma(2\nu+s+1)\Gamma(\nu+s+1)}, \quad \text{(A 11)}$$

where $x = \frac{1}{2}ik_0^2 l^2$. In obtaining (A 11), use has been made of the relation

$$\int_0^\infty \exp(-r^2/l^2)(r^2)^{\lambda+\nu+s} r\, dr = \tfrac{1}{2}l^2(l^2)^{\nu+\lambda+s}\Gamma(\nu+\lambda+s+1) \quad \text{(A 12)}$$

and

$$\Gamma(2y) = \frac{\Gamma(y)\Gamma(y+\frac{1}{2})}{2^{-2y+1}\sqrt{\pi}}. \quad \text{(A 13)}$$

Similarly it follows that

$$\int_0^\infty J_\nu(k_0 r) J_{-\nu}(k_0 r) \exp(-r^2/l^2) r^{2\lambda+1} dr = \frac{l^{2(\lambda+1)}}{2\sqrt{\pi}} \sum_{s=0}^\infty \frac{(2xi)^s \Gamma(s+\frac{1}{2})\Gamma(s+\lambda+1)}{s!\,\Gamma(s+\nu+1)\Gamma(s-\nu+1)}. \quad \text{(A 14)}$$

Equation (A 11) can be written in closed form by noting that (Nielsen 1904, p. 21)

$$J_\nu(x) e^{ix} = \frac{(2x)^\nu}{\sqrt{\pi}} \sum_{s=0}^\infty \frac{(2xi)^s \Gamma(\nu+s+\frac{1}{2})}{s!\,\Gamma(2\nu+s+1)}. \quad \text{(A 15)}$$

Combining (A 15) and (A 11) for the case when $\lambda = 0$ yields the (known) result that

$$\int_0^\infty J_\nu(k_0 r) J_\nu(k_0 r) \exp(-r^2/l^2) r\, dr = l^2/2\, J_\nu(x) e^{ix}(-i)^\nu. \quad \text{(A 16)}$$

Likewise it is easy to show that

$$\int_0^\infty J_\nu(k_0 r) J_\nu(k_0 r) \exp(-r^2/l^2) r^3 dr = \frac{l^4}{2}\frac{d}{dx}[x(-i)^\nu J_\nu(x) e^{ix}]. \quad \text{(A 17)}$$

The integrals involving $J_\nu J_{-\nu}$ can likewise be expressed in closed form. For integral values of n, J_n can be expressed as (Watson 1944)

$$J_n(x) = \frac{1}{\pi i^n}\int_0^\infty \exp(ix\cos\phi)\cos(n\phi)\, d\phi. \quad \text{(A 18)}$$

For arbitrary ν, let

$$\phi_\nu(x) = \frac{1}{\pi i^\nu}\int_0^\infty \exp(ix\cos\phi)\cos(\nu\phi)\, d\phi. \quad \text{(A 19)}$$

Clearly ϕ_ν reduces to J_n for integral ν. Furthermore, it can be shown that

$$e^{ix} i^\nu \phi_\nu(x) = \frac{1}{\sqrt{\pi}} \sum_{s=0}^\infty \frac{\Gamma(s+\frac{1}{2})(2xi)^s}{\Gamma(\nu+s+1)\Gamma(s-\nu+1)}. \quad \text{(A 20)}$$

The function ϕ_ν is closely related to the Neumann function $Y_n(x)$. In this regard Nielsen has shown that

$$Y_n(x) = \frac{2}{\pi}\frac{\partial J_\nu}{\partial \nu}\bigg|_{\nu=n} - 2\frac{\partial \phi_\nu}{\partial \nu}\bigg|_{\nu=n} - i\pi J_n(x) - \sigma_n(x), \quad \text{(A 21)}$$

where

$$\sigma_n(x) = 2\pi^{\frac{1}{2}} e^{-ix} i^n \sum_{s=0}^{n-1} \frac{(2n-s-1)!}{s!\,\Gamma(n-s+\frac{1}{2})(2xi)^{n-s}}. \quad \text{(A 22)}$$

It follows immediately from (A 21) and (A 14) that for $\lambda = 0$

$$\int_0^\infty J_\nu(k_0 r)\, J_{-\nu}(k_0 r) \exp(-r^2/l^2)\, r\, dr = \tfrac{1}{2} l^2 e^{ix} i^\nu \phi_\nu(x) \tag{A 23}$$

and, by analogy with (A 17),

$$\int_0^\infty J_\nu(k_0 r)\, J_{-\nu}(k_0 r) \exp(-r^2/l^2)\, r^3\, dr = \frac{l^4}{2}\frac{\partial}{\partial x}\{x(i)^\nu e^{ix} \phi_\nu(x)\}. \tag{A 24}$$

By definition, $$H_\nu(k_0 r) = J_\nu(k_0 r) + iY_\nu(k_0 r),$$

with $$Y_\nu(k_0 r) = [\cos(\nu\pi)\, J_\nu(k_0 r) - J_{-\nu}(k_0 r)]/\sin(\nu\pi).$$

Consequently

$$\int_0^\infty Y_\nu(k_0 r)\, J_\nu(k_0 r) \exp(-r^2/l^2)\, r\, dr = \tfrac{1}{2} l^2 e^{ix} e^{-\frac{1}{2}\nu\pi i} \left\{\frac{\cos(\nu\pi)\, J_\nu(x) - e^{\nu\pi i}\phi_\nu(x)}{\sin(\nu\pi)}\right\}.$$

Taking the limit as $\nu \to n$, applying l'Hôpital's rule and using (A 21) results in the relation

$$\int_0^\infty Y_n(k_0 r)\, J_n(k_0 r) \exp(-r^2/l^2)\, r\, dr = \tfrac{1}{2} l^2 e^{ix} e^{-\frac{1}{2}(n+1)\pi i} [\tfrac{1}{2} H_n^{(1)}(x) + (i/2\pi)\, \sigma_n(x)] \tag{A 25}$$

and, in exactly the same fashion,

$$\int_0^\infty Y_n(k_0 r)\, J_n(k_0 r) \exp(-r^2/l^2)\, r^3\, dr = \frac{l^4}{2}\frac{\partial}{\partial x}\left\{x\, e^{ix} e^{-\frac{1}{2}(n+1)\pi i}\left(\frac{H_n^{(1)}}{2} + \frac{i}{2\pi}\sigma_n\right)\right\}. \tag{A 26}$$

The expressions on the right-hand sides of the above equations can be rewritten in terms of the modified Bessel functions through the use of the relations (see, for example, Watson 1944, p. 77)

$$\left.\begin{aligned} I_\nu(z) &= (-i)^\nu J_\nu(iz) \\ K_\nu(z) &= \tfrac{1}{2}\pi i H_\nu^{(1)}(iz) \end{aligned}\right\} \quad (-\pi \leqslant \arg z \leqslant \tfrac{1}{2}\pi). \tag{A 27}$$

When this has been done, and the expressions resulting from differentiation have been simplified through the recursion relations for I_n, K_n, one obtains

$$\int_0^\infty H_0^{(1)}(k_0 r)\, J_0(k_0 r) \exp(-r^2/l^2)\, r\, dr = \tfrac{1}{2} l^2 \exp(-r)\left[I_0(r) - \frac{i}{\pi} K_0(r)\right], \tag{A 28}$$

$$\int_0^\infty H_1^{(1)}(k_0 r)\, J_1(k_0 r) \exp(-r^2/l^2)\, r\, dr = \tfrac{1}{2} l^2 \exp(-r)\left[I_1(r) + \frac{i}{\pi} K_1(r)\right] - \frac{i}{\pi k_0^2}, \tag{A 29}$$

$$\int_0^\infty H_1^{(1)}(k_0 r)\, J_1(k_0 r) \exp(-r^2/l^2)\, r^3\, dr = \tfrac{1}{2} l^4 \exp(-r)\left[r(I_0(r) - I_1(r)) - \frac{ir}{\pi}(K_0(r) + K_1(r))\right], \tag{A 30}$$

where $r = \frac{1}{2} k_0^2 l^2$. Substituting these expressions into the expression for $\bar{Q}(k_0)$ yields the following expression for k_e^2:

$$k_e^2 = k_0^2\left[1 + \epsilon^2\left(1 - \frac{1}{2r}\right)\right] + \frac{\epsilon^2 k_0^2 r}{2} \exp(-r)\left[i\pi\left(I_0(r) - \frac{1}{r} I_1(r)\right) + \left(K_0(r) + \frac{1}{r} K_1(r)\right)\right]. \tag{A 31}$$

REFERENCES

BROWN, W. P. 1966 *J. Opt. Soc. Am.* **56**, 1045–1062.

CAGLE, J. B. 1962 Tsunami. *Naval Res. Rev.* (*February* 1962), pp. 15–17.

COPSON, E. T. 1965 *Asymptotic Expansions*. Cambridge University Press.

FRISCH, U. 1968 *Wave Propagation in Random Media in Probabilistic Methods in Applied Mathematics*, vol. 1. Academic.

KAJIURA, K. 1961 *I.U.G.G. Monograph, Comptes Rendus*, **24**, 206–230.

KATZ, E. J. 1962 Ph.D. thesis, Johns Hopkins University.

KAY, I. & SILVERMAN, R. A. 1958 *Nuovo Cimento*, **9** (suppl.), 626–645.

KELLER, J. F. 1958 *J. Fluid Mech.* **4**, 607–614.

MCKENNA, J. & MORRISON, J. A. 1970 *J. Math. Phys.* **11**, 2348–2360.

MOLYNEUX, J. E. 1971 *J. Opt. Soc. Am.* **61**, 248–255.

NIELSEN, N. 1904 *Handbuch der Theorie der Cylinderfunctionen*. Leipzig: Tuebner.

SMITH, S. M., CHASE, T. E., FARRELL, W. E., TAYLOR, I. L. & WEED, M. E. 1965 Distribution and statistical analysis of sea-floor relief in the north-east Pacific. *Scripps Inst. Oceanography, University California, La Jolla.*

STOKER, J. J. 1957 *Water Waves*. Interscience.

WATSON, G. N. 1944 *Treatise on Theory of Bessel Functions*, 1st edn. Cambridge University Press.